TRIUMF

Constant Tune Cyclotrons

Baartman, December 2022

Outline

- Some history
- Conventional approach
- Reversed approach
- Theory
- Result for compact cyclotron
- Result for ring cyclotron

“Isochronous” Cyclotrons

Fundamental principle: length ($\mathcal{L} = 2\pi\mathcal{R}$) of a closed orbit for a particle of a particular energy (γmc^2) is strictly proportional to particle speed (βc) at that energy:

$$\mathcal{R} = \beta\mathcal{R}_\infty. \tag{1}$$

The average magnetic field drops out of this requirement:

$$\langle B\rangle = \frac{\oint B\,\mathrm{d}s}{2\pi\mathcal{R}} = \frac{\oint B\rho\,\mathrm{d}\theta}{2\pi\mathcal{R}} = \frac{B\rho}{\beta\mathcal{R}_\infty} := \frac{\beta\gamma B_\mathrm{c}\mathcal{R}_\infty}{\beta\mathcal{R}_\infty} = \gamma B_\mathrm{c}. \tag{2}$$

For constant field then we know the logarithmic field gradient (sometimes called index):

$$k = \frac{R}{B}\frac{\mathrm{d}B}{\mathrm{d}R} = \frac{\beta}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}\beta} = \beta^2\gamma^2. \tag{3}$$

Tunes for simple case

Radial:

$$\nu_x^2 = 1 + k = 1 + \beta^2\gamma^2 = \gamma^2 \tag{4}$$

(see [Symon et al., 1956] for tune as function of k), or

$$\nu_x = \gamma \tag{5}$$

Vertical:

$$\nu_y^2 = -k = -\beta^2\gamma^2 \tag{6}$$

or

$$\nu_y = \beta\gamma i \tag{7}$$

which is imaginary and we need strong focusing to overcome this gradient effect that is required by isochronism.

Traditionally...

Add flutter ('Thomas' focusing [Thomas, 1938]) and spiral angle strong focusing ('Kerst' focusing [Kerst et al., 1954]) to compensate for the unfortunate required gradient.

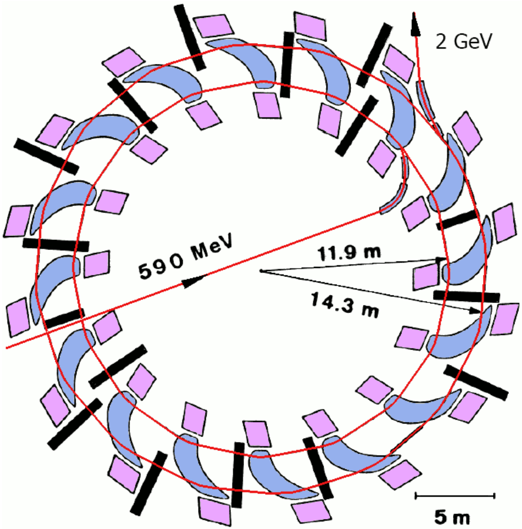
Traditionally, just enough was added to make the tune ν_y real and preferably constant. Just enough meant little change of the radial tune away from $\nu_x = \gamma$. Tune variation means betatron resonances crossed.

But! Can do better than that...

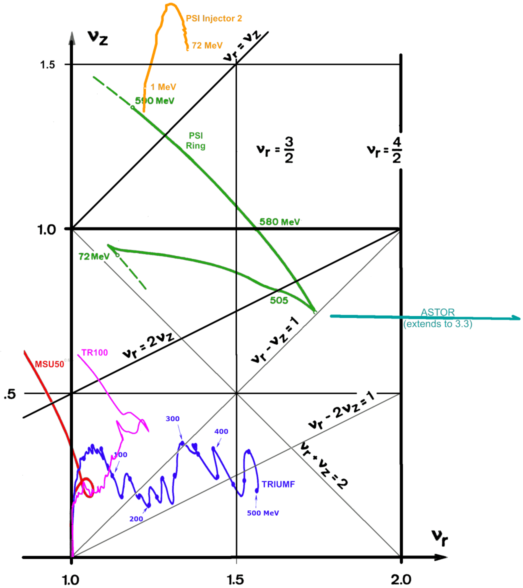
Cyclotron design can successfully traverse low order resonances if energy gain sufficiently high, but if tunes are fixed machine is less sensitive; can be built with wider error tolerance on magnetic field.

Proposed 2 GeV cyclotron (1983) never built

Proton cyclotron for 2 GeV, injecting at 590 MeV, devised by Werner Joho[Joho, 1983]. The magnets (light blue and pink) are similar to those that exist for the PSI 590 MeV cyclotron, which currently produces 2.4 mA protons.

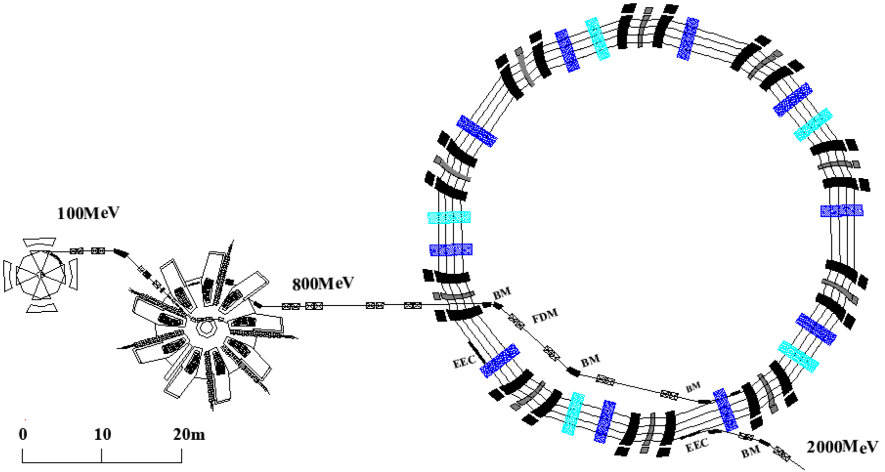


Betatron Resonances: Tune Diagram



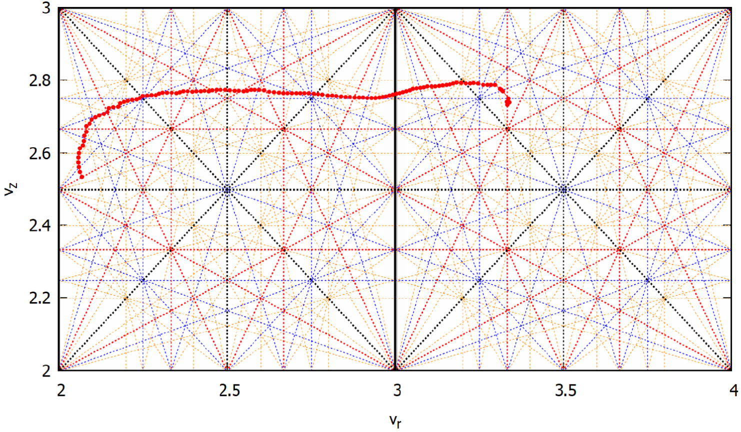
A sampling of machines' tunes displayed on a tune diagram. PSI 590 MeV ring cyclotron, the TRIUMF 500 MeV cyclotron, MSU's old 50 MeV proton cyclotron, Proposed TR100, PSI Injector 2, ASTOR. Both operating high energy machines (and RIKEN SRC?) traverse the $\nu_x = \frac{3}{2}$ resonance. The MSU and PSI machines traverse the Walkinshaw resonance $\nu_x = 2\nu_y$. ASTOR would cross $\nu_x = 2, \frac{5}{2}, 3$.

Proposed 2 GeV cyclotron (2019) not built yet



Proposed China Institute of Atomic Energy 2 GeV cyclotron [Zhang et al., 2019].

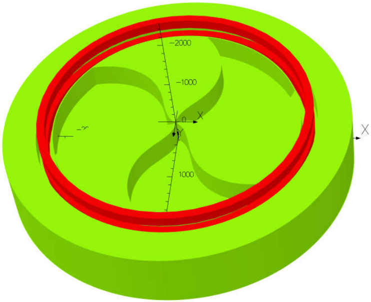
Proposed 2 GeV Cyclotron (2019) [Zhang et al., 2019]



Horizontal tune crosses integers in all the high-energy cyclotrons that have been proposed so far. Avoiding it comes at cost of compromised isochronism.

Cyclotron Design: Conventional Approach

Start from a magnet geometry, calculate the field distribution, track particles to **find the closed orbit**, compute the isochronous field error, the tunes, and iterate.



It can take days to produce one isochronous field distribution in this way: ability to explore is limited. Figure from Lige Zhang, TRIUMF.

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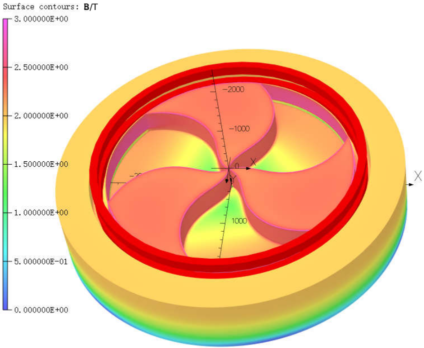

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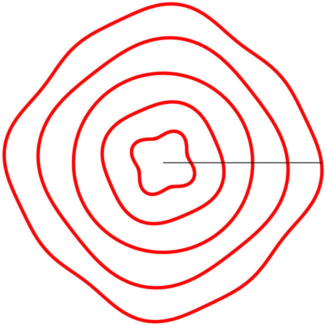
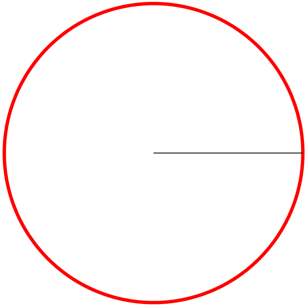
Reversed Approach: Starting from the Geometry of the Orbits

Synchrotrons are designed from the orbits why not cyclotrons? The difference is that synchrotrons only need **one** orbit. Cyclotrons need infinitely many!



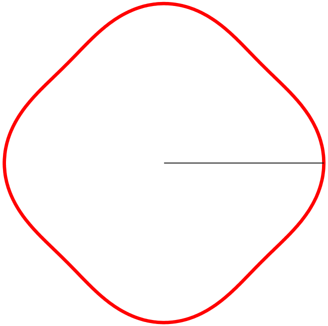
Ingredients

Circular orbit: $r(\theta) = a$



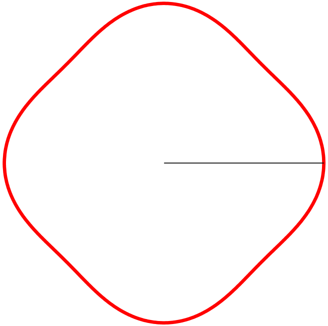
Ingredients

General **closed** orbit : $r(\theta) = r(\theta + 2\pi/N)$ with $N \in \mathbb{N}^*$



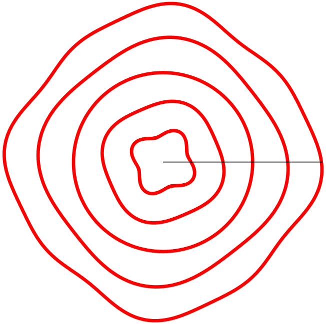
Ingredients

General **closed** orbit (Fourier series): $r(\theta) = \sum_{j=0}^{\infty} C_j \cos(j(\theta + \phi_j))$



Ingredients

Continuum of closed orbits: $r(a,\theta) = a\sum_{j=0}^{\infty} C_j(a)\cos(j(\theta+\phi_j(a)))$



a is the average radius of the orbit. For closed orbit to never cross: $\frac{\partial r}{\partial a} > 0$

Ingredients

Isochronous condition (cyclotron!):

$$\beta(a) = \frac{\mathcal{L}(a)}{2\pi\mathcal{R}_\infty}\,,$$

where $\mathcal{R}_\infty$ is a constant, $\beta(a)$ is the particle velocity and $\mathcal{L}(a)$ is the corresponding orbit circumference:

$$\mathcal{L}(a) = \int_0^{2\pi} \frac{\mathrm{d}s}{\mathrm{d}\theta}\,\mathrm{d}\theta\,,$$

$$\frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{r^2 + \left(\frac{\partial r}{\partial\theta}\right)^2}\,.$$

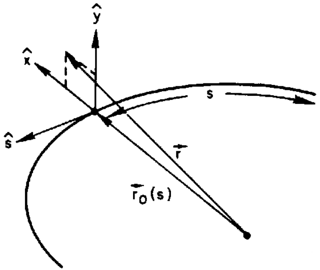
Objective

Given $r(a, \theta)$ + the isochronous condition, calculate the transverse tunes:

$$\nu_x(a)$$

$$\nu_y(a)$$

Frenet-Serret coordinate system (x, y, s)



Linear Motion around Closed orbit [Courant and Snyder, 1958]

ANNALS OF PHYSICS: **3,** 1–48 (1958)

Theory of the Alternating-Gradient Synchrotron*†

E. D. Courant and H. S. Snyder

Brookhaven National Laboratory, Upton, New York

The equations of motion of the particles in a synchrotron in which the field gradient index

$$n = -(r/B)\partial B/\partial r$$

varies along the equilibrium orbit are examined on the basis of the linear

Linear Motion around Closed orbit [Courant and Snyder, 1958]

The equations of motion are derived from the Hamiltonian

$$H = eV + c[m^2c^2 + (\mathbf{p} - e\mathbf{A})^2]^{1/2}, \tag{B9}$$

where V and $\mathbf{A}$ are the scalar and vector potentials of the electromagnetic field. In terms of the new variables this equals

$$\begin{aligned} H = eV + c\Big\{m^2c^2 + \frac{1}{(1+\Omega x)^2}[p_s - eA_s + \omega z(p_x - eA_x) \\ - \omega x(p_z - eA_z)]^2 + (p_x - eA_x)^2 + (p_z - eA_z)^2\Big\}^{1/2}, \end{aligned} \tag{B10}$$

...

The linearized equations of motion are obtained by expanding G as a power series in x, p_x, z, p_z and retaining only terms up to the second order. We consider a static magnetic field, so that $V = 0$ and $\mathbf{A}$ is independent of time. We may choose a gauge such that the power series expansions of the components of $\mathbf{A}$ are in the form

$$\begin{aligned} A_s &= ax + bz + cx^2 + dxz + ez^2 + \cdots, \\ A_x &= -fz + \cdots, \\ A_z &= fx + \cdots. \end{aligned} \tag{B15}$$

Linear Transverse Motion Hamiltonian in Cyclotrons

$$\mathcal{H} = \frac{x^2}{2}\frac{1-n}{\rho^2} + \frac{y^2}{2}\frac{n}{\rho^2} + \frac{p_x^2}{2} + \frac{p_y^2}{2}$$

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ρ, and n from geometry

$$\rho(a,\theta) = \frac{\left(r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2\right)^{3/2}}{r^2 + 2\left(\frac{\partial r}{\partial \theta}\right)^2 - r\frac{\partial^2 r}{\partial \theta^2}}$$

Remember:

$$r(a,\theta) = a\sum_{j=0}^{\infty} C_j(a)\cos(j(\theta + \phi_j(a)))$$

ρ and n from geometry

n can also be calculated entirely from the geometry of the orbits, but that is more tedious to demonstrate, for details see [Planche, 2019, Planche, 2022]

The field index n is obtained from:

$$n = -\frac{\rho}{B_0}\frac{\partial B}{\partial x} = \frac{\partial \rho}{\partial x} - \frac{\rho}{\beta\gamma}\frac{\partial \beta\gamma}{\partial x} \tag{2.14}$$

where $\beta\gamma = \frac{\beta}{\sqrt{1-\beta^2}}$ and β is given by Eq. (2.10). As shown in Ref. [11] the chain rule and the

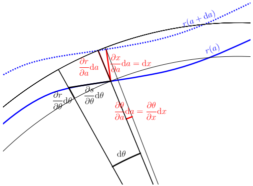


Figure 1. Relations between infinitesimal quantities around the closed orbit (the thick blue line); $\mathrm{d}\theta$ and $\mathrm{d}a$ represent infinitesimal variations in θ and a respectively.

relations between infinitesimal quantities illustrated in Fig. 3 yields:

$$\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial \rho}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{1}{r}\left(\frac{\partial \rho}{\partial a}\frac{\frac{\mathrm{d}s}{\mathrm{d}\theta}}{\frac{\partial r}{\partial a}} - \frac{\partial \rho}{\partial \theta}\frac{\frac{\partial r}{\partial \theta}}{\frac{\mathrm{d}s}{\mathrm{d}\theta}}\right), \tag{2.15}$$

Linear Transverse Motion Hamiltonian in Cyclotrons

What is important is that we now know the value of n and ρ along any orbit, entirely from geometry. We can calculate betatron tunes directly integrating the equations of motion from:

$$\mathcal{H} = \frac{x^2}{2}\frac{1-n}{\rho^2} + \frac{y^2}{2}\frac{n}{\rho^2} + \frac{p_x^2}{2} + \frac{p_y^2}{2}$$

using for instance Meade's algorithm [Meade, 1971].

To check the result with a more conventional tracking code one needs to produce a field map. Remembering $B\rho = \gamma m v/q$:

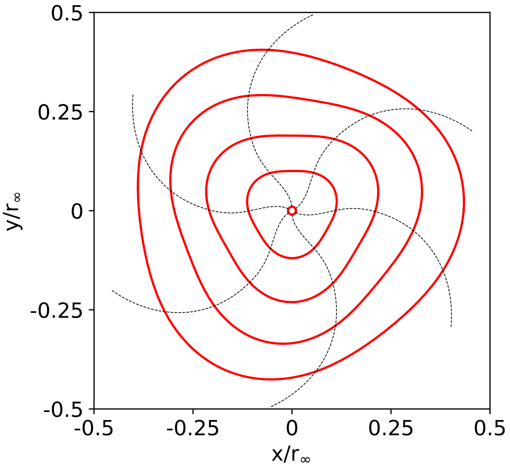
$$B(r,\theta) = \frac{\beta(a)}{\sqrt{1-\beta^2(a)}}\frac{m}{q\rho(a,\theta)}\ .$$

Flat Tunes Compact Cyclotron

$$r(a,\theta) = a\left(1 + C(a)\cos\left(N(\theta - \phi(a))\right)\right)$$

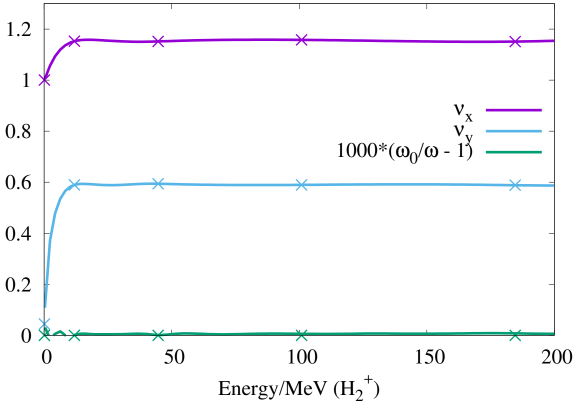
One now needs to find a way to define the two functions $C(a)$ and $\phi(a)$ using a finite – and hopefully small – number of degrees of freedom. We do this by constraining the values of $C(a)$ and $\phi(a)$ for a few orbits, and we use a cubic spline interpolation for any intermediate value of a.

Flat Tunes Compact Cyclotron

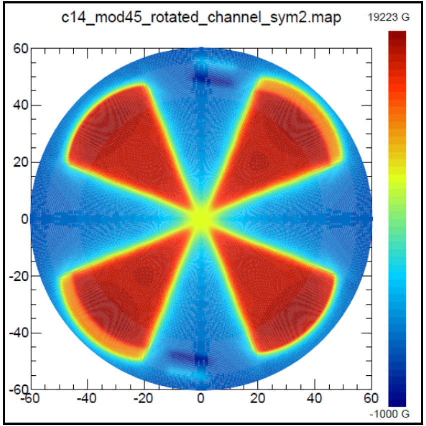


Quiz: So what did we do that was different? Large flutter at centre!

Flat Tunes Compact Cyclotron



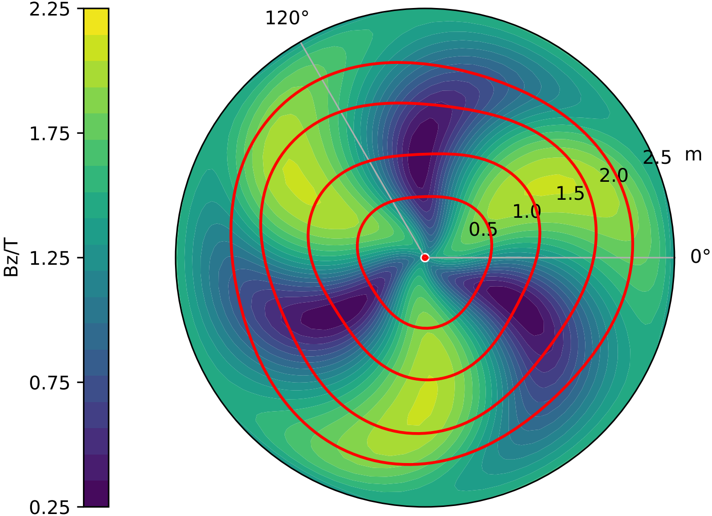
Central Orbit: Circular!



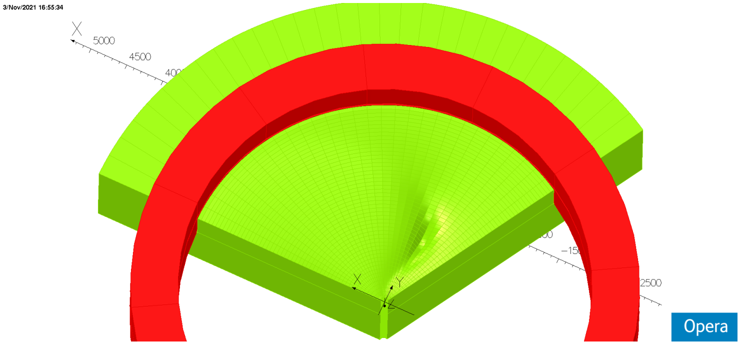
Courtesy of Wiel Kleeven, IBA.

Not quite constant because: Central orbit is circular!

Can one design a magnet to produce such a field?



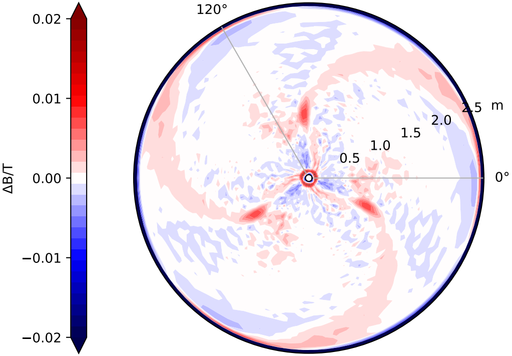
Can one design a magnet to produce such a field?



View of the magnet steel (green) and coils (red) as implemented in the 3D model.
Note that only 1/6 of the magnet is shown here (symmetry).

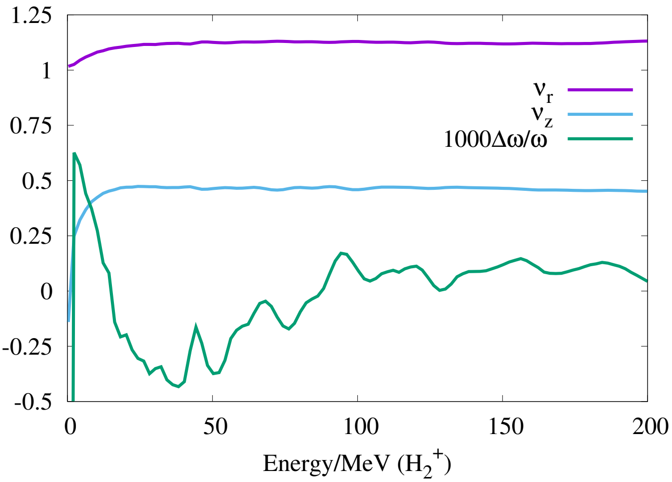
Notice the dimple.

Can one design a magnet to produce such a field?

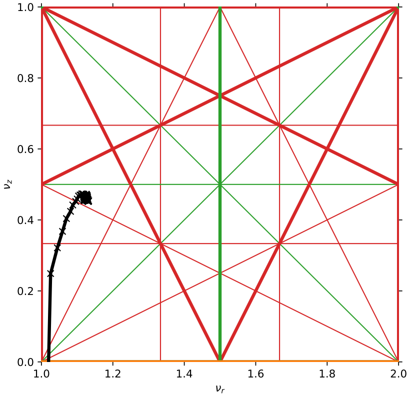


Difference between the desired field distribution, and the result from the 3-dimensional magnet model (finite element code: OPERA).

Tune Variation from 3D Magnet Model

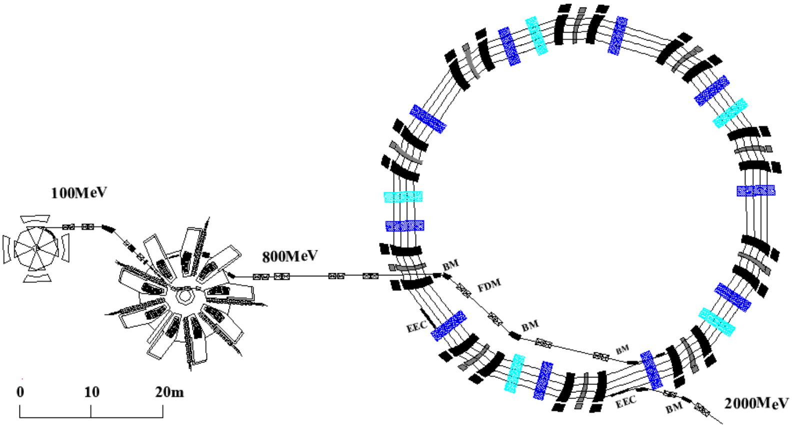


Tune Variation from 3D Magnet Model



What about a high-energy cylotron?

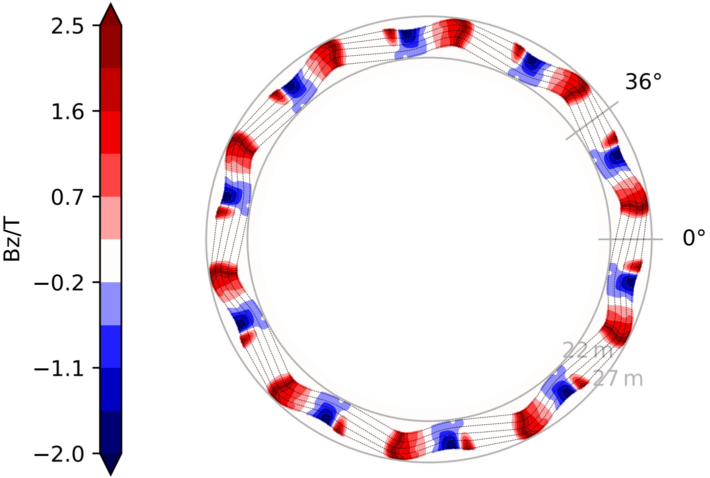
Using the same parameters as for the CIAE 2 GeV proton cyclotron design showed earlier:



Constant-tune 2 GeV Cyclotron

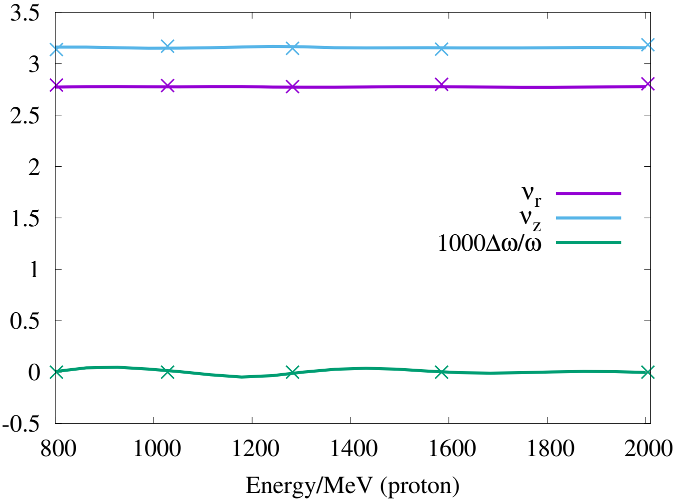
It took a rather more complex parametrization of the orbits, especially to get the straight sections right. Need at least 5 Fourier harmonics to have “field-free” regions. In the end it uses 25 free parameters. It took many iterations, and on the order of half a day on Thomas’ laptop to converge to this:

Constant-tune 2 GeV Cyclotron $B(r, \theta)$

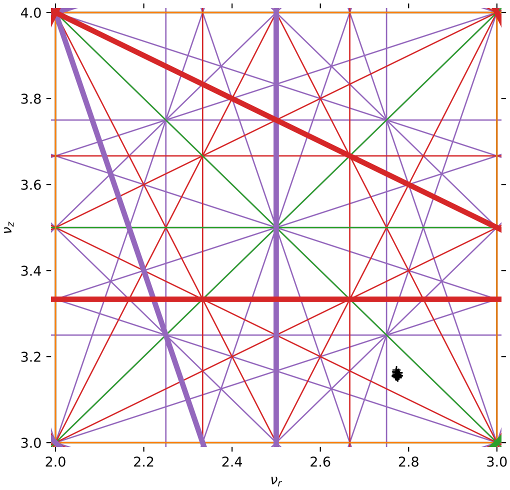


What features do you notice?

Constant-tune 2 GeV Cyclotron



Constant-tune 2 GeV Cyclotron



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Thank You

Thank you for your attention